Partitioning of Positive Integers

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ABSTRACT

In this paper, we give a brief introduction to the problem of partitioning positive integers as well as a survey of the most important work from Euler to Ono. In our results, we state a basic algorithm that determines all possible partitions of a positive integer. A computer algorithm (code) is provided as well as many examples. Two simple algebraic formulas that determine all possible partitions of a positive integer using the numbers 1 and 2, and 1, 2 and 3, are provided. Last but not least, we draw a conclusion from our results that the number of partitions of any positive integer can be written as a summation of a number of integers that are partitioned into ones and twos. These numbers, however, appear according to a specific pattern.

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INTRODUCTION

1.1 Definition of the problem

A partition of a positive integer n is an expression of n as a sum of positive integers. Partitions are considered the same if the summands differ only by order. Let $P(n)$ be the number of partitions of n. By convention, we define $P(0) = 1$.

1.2 Example

1 can be written as 1, hence $P(1) = 1$.

2 can be written as 1+1 and 2, hence $P(2) = 2$.

3 can be written as 1+1+1, 1+2 and 3, here 1+2 is equivalent to 2+1 hence it is only counted once, therefore $P(3) = 3$.

4 can be written as a 1+1+1+1, 1+1+2, 1+3, 2+2 and 4, hence $P(4) = 5$.

1.3 Motivation

My motivation behind working on the partitioning of positive integers started when Dr. Ayman Badawi asked in the abstract algebra course in the spring of 2012 the following question: "Who can write a computer program that determines all non-isomorphic abelian groups of order P^n?" I suddenly raised my hand and I told him “this is simple professor, I can do it and hand it in to you tomorrow.” He replied back “do you think this is easy?” I told him “for me it sounds simple.” He then told me “try it, but if it takes a lot of your time, then leave it for another day, perhaps summer break, just don’t spend too much time on it." Then briefly, Prof. Badawi introduced the work of Ramanujan on partitioning positive integers. Prof. Badawi stated that in order for one to answer the question one needs to calculate $P(n)$. Considering P^5, I obtained the following 7 non-isomorphic abelian groups of order P^5:

$$\mathbb{Z}_{p^1} \oplus \mathbb{Z}_{p^1} \oplus \mathbb{Z}_{p^1} \oplus \mathbb{Z}_{p^1} \oplus \mathbb{Z}_{p^1}$$

$$\mathbb{Z}_{p^1} \oplus \mathbb{Z}_{p^1} \oplus \mathbb{Z}_{p^1} \oplus \mathbb{Z}_{p^2}$$

$$\mathbb{Z}_{p^1} \oplus \mathbb{Z}_{p^1} \oplus \mathbb{Z}_{p^3}$$

$$\mathbb{Z}_{p^1} \oplus \mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2}$$

$$\mathbb{Z}_{p^1} \oplus \mathbb{Z}_{p^4}$$

$$\mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^3}$$

$$\mathbb{Z}_{p^5}$$

The numbers of non-isomorphic abelian groups of order $P(n)$ where $1 \leq n \leq 10$ are indeed $P(1) = 1,\ P(2) = 2,\ P(3) = 3,\ P(4) = 5,\ P(5) = 7,\ P(6) = 11,\ P(7) = 15,\ P(8) = 22,\ P(9) = 30,\ P(10) = 42.$

The subject of partitioning positive integers respectively is very rich and deep. Covering every aspect of partitions would take thousands of pages. Section 2 in this paper provides a quick overview of partitions (from Euler to Ono), introduces a few techniques for dealing with partitions, and explores some interesting problems. This paper will hopefully shed some light on the beauty of partitions, combinatorics, and mathematics in general. In section 3, we present our work on partitioning positive integers.

1.4 Different representations of the partitions

A partition can be written with a graphical representation, known as a Ferres graph, named after the British mathematician Norman Macleod Ferres, (see, for example [1] for details). For example 5 = 2 + 2 + 1 can be written as:

2	2	1
•	•	•
•	•	

The conjugate graph is obtained from a graph by writing the columns as rows. Hence 5=3+2 can be drawn as:

3	2
•	•
•	•
•	

The term **Conjugate** in partitioning means reflecting the dots across the diagonal, that is, replacing every column by a row.

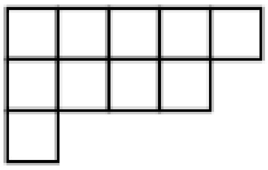
If the conjugate of a partition happens to be the same as itself, we call this partition as **self-conjugate.**

Hence the conjugate of (2,2,1) is (3,2).

An example of a self- conjugate partition is (4,2,1,1) for the number 8, which looks as follows:

$$\begin{array}{ccccc} & 4 & 2 & 1 & 1 \\ 4 & \bullet & \bullet & \bullet & \bullet \\ 2 & \bullet & \bullet & & \\ 1 & \bullet & & & \\ 1 & \bullet & & & \end{array}$$

Alfred Young developed the Young Diagram for partitioning positive integers, which is similar to the Ferres diagram where the dots are replaced by boxes instead, such as:



which represents the partition of 10 into (5,4,1).
This is equivalent to the Ferres Diagram:

$$\begin{array}{ccc} 5 & 4 & 1 \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & \bullet & \\ \bullet & \bullet & \\ \bullet & & \end{array}$$

Although Young diagrams are similar to Ferres Diagrams, Young diagrams turn out to be extremely useful in the study of symmetric functions and group representation theory [2].

HISTORY OF THE PROBLEM

Although Euler is the first person to work on the problem of partitioning positive integers, Andrews in [3] explains how Leibniz, a German mathematician and philosopher and known as the father of calculus, was the first person to consider partitioning. Leibniz asked Bernoulli in a 1674 letter about the number of “divulsions” of integers, which means the number of partitions of integers in the modern terminology. Leibniz went on to partition several integers up till 6 where he suggested that the number of partitions of any number would always be prime. But reaching number 7, he found out that 7 has 15 partitions, hence his suggestions was wrong and the problem was open since then.

2.1 Leonhard Euler [1707 – 1783]

Leonhard Euler is a Swiss mathematician and physicist. In 1741, Euler gave a presentation on partitions of integers to the St. Petersburg Academy, which led to the first publication on partition of integers [4]. Euler discovered several theorems in the field of integer partitioning [5]. One of his greatest discoveries in this field was Euler’s generating function for integer partitioning which is given by:

$$\sum_{n=0}^{\infty} p(n)x^n = \frac{1}{\prod_{p=1}^{\infty} 1-x^p} \text{ , where } |x| < 1$$

Equation 1: Euler's Generating Function

For example, to find the number of partitions of 5, we extract the coefficient of x^5 from the right hand side of Equation 1:

$$\left(1 + x^1 + x^{1(2)} + x^{1(3)} + \cdots\right)\left(1 + x^2 + x^{2(2)} + x^{2(3)} + \cdots\right)\left(1 + x^3 + x^{3(2)} + x^{3(3)} + \cdots\right)\left(1 + x^4 + x^{4(2)} + x^{4(3)} + \cdots\right)\left(1 + x^5 + x^{5(2)} + x^{5(3)} + \cdots\right) \ldots$$

Equation 2: Expansion of Euler's Generating Function

The number of partitions of 5 according to Euler’s formula is the coefficient of x^5. Multiplying the polynomials of Equation 2, we get the following polynomial:

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + \cdots$$

Hence the number of partitions of 5 is 7.

Another interesting theorem proved by Euler, which ranks 16 in Wells’ list of the most beautiful theorems [6], states that the number of partitions of a positive integer n into odd-parts equals the number of partitions of n into distinct-parts.

Definition:

(1) Odd parts: The partition of any positive integer n, which comprises odd numbers only.

(2) Distinct Parts: The partition of any positive integer n, which contains no repeated numbers, that is no integer will appear twice in the same partition.

Let n be a positive integer. The number of partitions of n into odd parts equals the number of partitions of n into distinct parts.

Example:

4 can be written as

(1,1,1,1): odd part

(1,1,2)

(1,3): odd part and distinct part

(2,2)

(4): distinct part

Hence we have two distinct parts and two odd parts, that is, number of odd parts = number of distinct parts.

2.2 Percy Alexander MacMahon [1854 – 1929]

MacMahom is an English mathematician who is noted for his passion and contribution to the field of partitions of numbers and enumerative combinatorics. A paper published by University of Iowa in 2001 [7], mentioned that MacMahon computed the values of $p(n)$ for $n = 1,2,3, \ldots ,200,$ by hand which turned out to be immensely useful for Hardy and Ramanujan for checking the accuracy of their formula for approximating p(n). MacMohan found that

$p(200) = 3,972,999,029,388.$

2.3 Srinivasa Ramanujan [1887 – 1920]

Srinivasa Ramanujan is an Indian mathematician and autodidact in the field of pure mathematics, and made marvelous contributions to mathematical analysis, number theory, infinite series, and continued fractions. Ramanujan along with Godfrey Harold Hardy were able to find a non-convergent asymptotic series that permits exact computation of the number of partitions of an integer [8].

2.4 Godfrey Harold Hardy [1877 – 1947]

Godfrey Hardy is an English mathematician who had great achievements in the field of number theory and mathematical analysis. In 1918, Hardy and Ramanujan used the circle method and modular functions to obtain an asymptotic solution as given in [9]:

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} \quad \text{as } n \to \infty$$

Equation 3: Hardy-Ramanujan's Asymptotic Expression

For example, if we plug 50 and 70 into the equation we obtain the following results:

Table 1: Comparing Hardy-Ramanujan's Asymptotic Expression against the p(n) function

$\boldsymbol{n}$	$\boldsymbol{p(n)}$	$\boldsymbol{p(n)}$ **using** Equation 3	$\frac{\boldsymbol{p(n)}\ \textbf{using}\ \text{Equation 3}}{\boldsymbol{p(n)}}$
10	42	48.10430882	1.145340686
50	204226	217590.4992	1.065439754
70	4087968	4312669.963	1.054966664
80	15796476	16606781.57	1.051296604
. . .	. . .	. . .	. . .
∞	∞	∞	1

Hardy and Ramanujan made several discoveries in the same field and came up with several results as mentioned in [9], such as proving Ramanujan's congruence theorem:

For every n we have

$$p(5n+4) \equiv 0 \ (mod\ 5),$$
$$p(7n+5) \equiv 0 \ (mod\ 7),$$
$$\text{and } p(11n+6) \equiv 0 \ (mod\ 11).$$

Equation 4: Ramanujan's Congruences

For example, $p(5+4) = 30 \equiv 0\ (mod\ 5)$

Further work by Hardy and Ramanujan [2] made resulted in the following asymptotic expansion:

$$p(n) = \frac{1}{2\sqrt{2}} \sum_{k=1}^{v} A_k(n)\sqrt{k}\,\frac{d}{dn} e^{\pi\sqrt{\frac{2}{3}}\frac{\sqrt{\left(n-\frac{1}{24}\right)}}{k}}$$

$$\text{where} \quad A_k(n) = \sum_{\substack{o \le m < k \\ (m,k)=1}} e^{\pi i [s(m,k) - \frac{1}{k} 2nm]}$$

Equation 5: Hardy-Ramanujan's Asymptotic Expansion

2.5 George Neville Watson [1886 – 1965]

George Watson is an English mathematician who worked in the fields of complex analysis and the theory of special functions. Watson contributed to the problem of partitioning positive integers by proving several q-series identities. Due to the complexity of the identities and the limitation of the paper, the identities can be checked at [10] and [11]. Watson has also came up with partition congruences for powers of 7 and an alternative simple proof is in [12].

2.6 Hans Rademacher [1892 – 1969]

Hans Rademacher is a German Mathematician known for his research in mathematical analysis and number theory. Rademacher worked on Hardy-Ramanujan's Asymptotic Expansion (Equation 5), and in 1937 he obtained the final explicit form as explained in [9]:

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k} \left[\frac{d}{dx} \frac{\sinh\left(\frac{\pi}{k} \sqrt{\frac{2}{3}\left(x - \frac{1}{24}\right)} \right)}{\sqrt{x - \frac{1}{24}}} \right]_{x=n}$$

$$\text{where} \quad A_k(n) = \sum_{\substack{h \bmod k \\ (h,k)=1}} \omega_{h,k} e^{-2\pi i n h/k}$$

$\omega_{h,k}$ is a 24kth root of unity defined as follows:

$$\omega_{h,k} = \begin{cases} \left(\frac{-k}{h}\right) e^{-\pi i \left[\frac{1}{4}(2-kh-h) + \frac{1}{12}(k-k^{-1})(2h-h^* + h^2 h^*)\right]} & \text{if h is odd} \\ \left(\frac{-k}{h}\right) e^{-\pi i \left[\frac{1}{4}(k-1) + \frac{1}{12}(k-k^{-1})(2h-h^* + h^2 h^*)\right]} & \text{if h is even} \end{cases}$$

$$hh^* \equiv -1 (\bmod k)$$

$$\left(\frac{a}{b}\right) \text{is the Jacobi} - \text{Legendre sym.} = \begin{cases} 0 & \text{if } a = 0 \ (\bmod b) \\ 1 & \text{if } 0 \ne a = x^2 \ (\bmod b), \text{for some } x \\ -1 & \text{otherwise} \end{cases}$$

Equation 6: Hardy-Ramanujan-Rademacher's Formula

As mentioned above, MacMahon previously computed $P(200) = 3{,}972{,}999{,}029{,}388$. Using Equation 6 to compute the first 8 terms in the series, we get $P(200)$ to be:

$$
\begin{array}{lr}
 & 3{,}972{,}998{,}993{,}185.896 \\
+ & 36{,}282.978 \\
- & 87.555 \\
+ & 5.147 \\
+ & 1.424 \\
+ & 0.071 \\
+ & 0.000 \\
+ & 0.043 \\
\hline
 & 3{,}972{,}999{,}029{,}338.004
\end{array}
$$

which is equal to p(200) within 0.004.

The proof of Rademacher's formula involves Ford circles, Farey sequences, modular symmetry and the Dedekind eta function in a central way as indicated in [2].

2.7 Arthur Oliver Lonsdale Atkin [1925 – 2008]

Atkin is a British mathematician. He was one of the first mathematician to use computers to do research in pure mathematics. Along with Joseph Lehner, Atkin came up with the modula form that helped Andrew wiles to prove Fermat's Last Theorem, the problem that was open for over 300 years [13].

Atkin attempted solving several problems in partitioning integers, and in 1966 Atkin proved the following [14]:

(1) Suppose $l \equiv 4 \pmod 5$ is prime and $n \in Z^+$ with $l \nmid n$. If $n \equiv 23l \pmod{120}$ or $n \equiv 47l \pmod{120}$, then

$$p\left(\frac{l^3 n + 1}{24}\right) \equiv 0 \pmod 5.$$

(2) Suppose $l \equiv 3 \pmod 5$ is a prime exceeding 3 and $n \in Z^+$ with $\left(-\frac{n}{l}\right) = -1$. If $n \equiv 23 \pmod{120}$ or $n \equiv 47 \pmod{120}$, then

$$p\left(\frac{l^2 n + 1}{24}\right) \equiv 0 \pmod 5.$$

2.8 Freeman Dyson [1923 – Present]

Dyson is a British theoretical physicist and mathematician, famous for his work in several fields in physics such as astronomy and nuclear engineering. In 1944, Dyson discovered a combinatorial reason for the existence of the first two congruences and conjectured the existence of a crank function for partitions that would provide a combinatorial proof of Ramanujan's congruence modulo 11 (Equation 4). George Andrews and Frank Garvan successfully found such a function, and proved the celebrated result that the crank simultaneously "explains" the three Ramanujan congruences modulo 5, 7 and 11 [15]. More information about Dyson's conjecture is available in [16].

In addition, Dyson defined the rank function of a partition as the largest part minus the number of parts, which is explained in detail in [17]. He let *N(m,t,n)* denote the number of partitions of *n* of rank congruent to *m* modulo *t*, and he conjectured the following:

$$N(m,5,5n+4) = \frac{1}{5}p(5n+4),\quad 0 \le m \le 4$$

$$\text{and } N(m,7,7n+5) = \tfrac{1}{7}p(7n+5),\quad 0 \le m \le 6.$$

Equation 7: Dyson's Conjectures

2.9 George Andrews [1938 – Present]

George Andrews is an American mathematician working in Analysis and combinatorics. His famous book, "The Theory of Partitions" [9], was published in 1976. It contains a survey of the problem of partitions, and all the contributions to this field by previous mathematician and a detail of their approaches and latest theorems and results. Andrews is also famous for collecting Ramanujan's lost notes, and finding the crank function, which is proven in [17].

More information about Andrews motivational story can be found in [18].

2.10 ken Ono [1968 – Present]

Ken Ono is an American Mathematician who specializes in number theory and especially in integer partitions and modular forms. In 2011, Ono along with Jan Bruinier, published a paper discovering an algebraic formula for the partition function. Due to the limitation and scope of this paper, and due to the complexity of Ono's formula, details will not be disclosed here, and they can be reached in [19].

Another paper published by Folsom, Kent and Ono in 2012 shows a short proof of their result and is found in [20].

RESULTS

The following notations will be used through out this section.

If $m < n$, where $m, n \in \mathbb{Z}^{+}$, then we define

$\mathbf{P_m(n) :=}$ **the number of partitions of n,**
where m appears at least once in the numbers of each part and the other numbers are necessarily natural numbers $\mathbf{\leq m}$

$\mathbf{f_m(n) :=}$ **the number of partitions of n using only natural numbers** $\mathbf{\leq m}$

If $m > n$, then $P_m(n) = 0$, and $f_m(n) = f_n(n)$

For every positive integer m, we let $f_m(0) = 1$.

It is clear that

$$\boldsymbol{f_m(n) = P_m(n) + P_{m-1}(n) + P_{m-2}(n) + \cdots + P_1(n)}$$

Equation 8

Now let $2 \leq m \leq n$, where $m, n \in \mathbb{Z}^{+}$

$$n = qm + r, where\ q, r\ \in \mathbb{Z}^{+}\ and\ 0 \leq r < m$$

It should be clear that

$$\boldsymbol{P_m(n) = f_{m-1}(n-m) + f_{m-1}(n-2m) + f_{m-1}(n-3m) + \cdots + f_{m-1}(n-qm)}$$

Equation 9

Note that, $f_{m-1}(n - im)$ equals to the number of all partitions of n where m appears in each partition exactly *i* times and all other numbers of each partition are strictly less than m.

Note that $q = \left\lfloor \frac{n}{m} \right\rfloor$, hence in light of Equation 8 and Equation 9,

$f_m(n) = P_m(n) + f_{m-1}(n)$

$$\boldsymbol{f_m(n) = \sum_{i=1}^{\left\lfloor \frac{n}{m} \right\rfloor} f_{m-1}(n-im) + f_{m-1}(n) = \sum_{i=0}^{\left\lfloor \frac{n}{m} \right\rfloor} f_{m-1}(n-im)}$$

Equation 10

In particular if $n = m$, then

$$\boldsymbol{P(n) = f_n(n) = P_n(n) + f_{n-1}(n) = 1 + f_{n-1}(n)}$$

Equation 11

Equation 8, Equation 9, Equation 10 and Equation 11 describes our method of partitioning a positive integer n. See Section 3.3 for examples.

In the following result, we give an explicit formula for $f_2(n)$.

3.1 Proposition 1: An explicit formula for $\boldsymbol{f_2(n)}$

By definition, it is clear that $f_1(n) = P_1(n) = 1$.

Now let $n \in \mathbb{Z}^+$, then

$$f_2(n) = \left\lceil \frac{n+1}{2} \right\rceil$$

Equation 12

Proof

Let $m = 2, n = 2q + r,$ where $q, r \in \mathbb{Z}^+$ and $0 \leq r < 2$

$P_2(n) = f_1(n-2) + f_1(n-(2+2)) + f_1(n-(2+2+2)) \ldots f_1(n-2q)$

Note that $P_2(1) = 0$, hence assume that $n \geq 2$.

$f_2(n) = P_2(n) + P_1(n) = q + 1 = \left\lfloor \frac{n}{2} \right\rfloor + 1 = \left\lceil \frac{n+1}{2} \right\rceil$.

It is known that $f_3(n)$ equals to the nearest integer to $\frac{(n+3)^2}{12}$. In the following result, we give an explicit formula for $f_3(n)$.

3.2 Proposition 2: An explicit formula for $f_3(n)$

The number of partitions of any integer n into parts containing 1,2 and 3 is:

$$f_3(n) = (2a+1)\left\lceil\frac{n+1}{2}\right\rceil - \frac{3}{2}\left(\lfloor a\rfloor^2 + \lfloor a\rfloor + \lceil a\rceil^2 - \frac{1}{3}\lceil a\rceil\right) - (1 - |n\ (\mathrm{mod} 2)|)\frac{(\lceil a\rceil^2 + \lceil a\rceil)}{2} \quad \text{where } a = \frac{\left\lfloor\frac{n}{3}\right\rfloor}{2}$$

Equation 13

Proof

Let $m = 3, n = 3q + r,$ where $q, r \in \mathbb{Z}^+$ and $0 \le r < 3$

$f_3(n) = P_3(n) + f_2(n)$

$P_3(n) = f_2(n-3) + f_2(n-6) + f_2(n-9) + f_2(n-12) + f_2(n-15) + \cdots + f_2(n-3q)$

$$= \left\lceil\frac{(n-3)+1}{2}\right\rceil + \left\lceil\frac{(n-6)+1}{2}\right\rceil + \left\lceil\frac{(n-9)+1}{2}\right\rceil + \left\lceil\frac{(n-12)+1}{2}\right\rceil + \left\lceil\frac{(n-15)+1}{2}\right\rceil + \cdots + \left\lceil\frac{(n-3q)+1}{2}\right\rceil$$

$$= \left\lceil\frac{n+1}{2} - \frac{1}{2}\right\rceil - 3(0) - 1 + \left\lceil\frac{n+1}{2}\right\rceil - 3(1) + \left\lceil\frac{n+1}{2} - \frac{1}{2}\right\rceil - 3(1) - 1 + \left\lceil\frac{n+1}{2}\right\rceil - 3(2) + \left\lceil\frac{n+1}{2} - \frac{1}{2}\right\rceil - 3(2) - 1 + \cdots + \left\lceil\frac{n+1}{2} - \frac{3q}{2}\right\rceil$$

Equation 14

Consider the two cases, (a) when n is odd, and (b) when n is even:

(a) if n is odd

$$\left\lceil \frac{n+1}{2} - \frac{1}{2} \right\rceil = \left\lceil \frac{n+1}{2} \right\rceil$$

Hence

$$P_3(n) = \left\lceil \frac{n+1}{2} \right\rceil - 3(0) - 1 + \left\lceil \frac{n+1}{2} \right\rceil - 3(1) + \left\lceil \frac{n+1}{2} \right\rceil - 3(1) - 1 + \left\lceil \frac{n+1}{2} \right\rceil - 3(2) + \left\lceil \frac{n+1}{2} \right\rceil - 3(2) - 1 + \cdots + \left\lceil \frac{n+1}{2} - \frac{3q}{2} \right\rceil$$

We have several terms in the formula above, but the repeated term is $\left\lceil \frac{n+1}{2} \right\rceil$ and repeats q times. Hence $P_3(n) = q \left\lceil \frac{n+1}{2} \right\rceil$

$$\begin{aligned} &-3(0) - 1 + \\ &-3(1) + \\ &-3(1) - 1 + \\ &-3(2) + \\ &-3(2) - 1 + \\ &\ldots \end{aligned}$$

Equation 15

Considering the odd terms of Equation 15, we realize the same pattern. Rearranging the terms we get:

$$\begin{aligned} &-3(1) + 2 + \\ &-3(2) + 2 + \\ &-3(3) + 2 + \\ &\ldots \end{aligned}$$

and we get the following series:

$$\sum_{i=1}^{\left\lceil \frac{q}{2} \right\rceil} -3i + 2 = \left[\frac{-3}{2} \left(\left\lceil \frac{q}{2} \right\rceil \right) \left(\left\lceil \frac{q}{2} \right\rceil + 1 \right) + 2 \left\lceil \frac{q}{2} \right\rceil \right]$$

Equation 16

Considering the even terms of the sequence in Equation 15:

$$\begin{aligned} &-3(1) + \\ &-3(2) + \cdots \end{aligned}$$

And we get the following series:

$$-3\sum_{i=1}^{\left\lfloor\frac{q}{3}\right\rfloor} i = -3\left[\frac{1}{2}\left(\left\lfloor\frac{q}{2}\right\rfloor\right)\left(\left\lfloor\frac{q}{2}\right\rfloor+1\right)\right]$$

Equation 17

Combining the first term in Equation 15 with Equation 16 and Equation 17, we get:

$$P_3(n) = n_3\left\lceil\frac{n+1}{2}\right\rceil + \left[\frac{-3}{2}\left(\left\lceil\frac{q}{2}\right\rceil\right)\left(\left\lceil\frac{q}{2}\right\rceil+1\right)+2\left\lceil\frac{q}{2}\right\rceil\right] + -3\left[\frac{1}{2}\left(\left\lfloor\frac{q}{2}\right\rfloor\right)\left(\left\lfloor\frac{q}{2}\right\rfloor+1\right)\right]$$

$$P_3(n) = n_3\left\lceil\frac{n+1}{2}\right\rceil + \left[\frac{-3}{2}\left(\left\lceil\frac{q}{2}\right\rceil\right)\left(\left\lceil\frac{q}{2}\right\rceil+1\right)+2\left\lceil\frac{q}{2}\right\rceil\right] + \left[\frac{-3}{2}\left(\left\lfloor\frac{q}{2}\right\rfloor\right)\left(\left\lfloor\frac{q}{2}\right\rfloor+1\right)\right]$$

$$P_3(n) = n_3\left\lceil\frac{n+1}{2}\right\rceil + \frac{-3}{2}\left(\left\lceil\frac{q}{2}\right\rceil\right)\left(\left\lceil\frac{q}{2}\right\rceil+1\right)+2\left\lceil\frac{q}{2}\right\rceil + \frac{-3}{2}\left(\left\lfloor\frac{q}{2}\right\rfloor\right)\left(\left\lfloor\frac{q}{2}\right\rfloor+1\right)$$

$$P_3(n) = n_3\left\lceil\frac{n+1}{2}\right\rceil + \frac{-3}{2}\left\lceil\frac{q}{2}\right\rceil^2 + \frac{-3}{2}\left\lceil\frac{q}{2}\right\rceil + 2\left\lceil\frac{q}{2}\right\rceil + \frac{-3}{2}\left\lfloor\frac{q}{2}\right\rfloor^2 + \frac{-3}{2}\left\lfloor\frac{q}{2}\right\rfloor$$

$$P_3(n) = n_3\left\lceil\frac{n+1}{2}\right\rceil - \frac{3}{2}\left\lfloor\frac{q}{2}\right\rfloor^2 - \frac{3}{2}\left\lfloor\frac{q}{2}\right\rfloor - \frac{3}{2}\left\lceil\frac{q}{2}\right\rceil^2 - \frac{3}{2}\left\lceil\frac{q}{2}\right\rceil + 2\left\lceil\frac{q}{2}\right\rceil$$

Equation 18

(b) if n is even

$$\left\lceil\frac{n+1}{2}-\frac{1}{2}\right\rceil = \left\lceil\frac{n+1}{2}\right\rceil - 1$$

Hence substitute $\left\lceil\frac{n+1}{2}\right\rceil - 1$ in Equation 14. We get:

$$P_3(n) = \left\lceil\frac{n+1}{2}\right\rceil - 1 - 3(0) - 1 + \left\lceil\frac{(n+1)}{2}\right\rceil - 3(1) + \left\lceil\frac{n+1}{2}\right\rceil - 1 - 3(1) - 1$$

$$+ \left\lceil\frac{(n+1)}{2}\right\rceil - 3(2) + \left\lceil\frac{n+1}{2}\right\rceil - 1 - 3(2) - 1 + \cdots$$

$$+ \left\lceil\frac{(n+1)}{2} - \frac{3q}{2}\right\rceil$$

After arranging the above equation we get:

$$P_3(n) = \left\lceil\frac{n+1}{2}\right\rceil - 1 - 3(0) - 1 +$$

$$\left\lceil\frac{n+1}{2}\right\rceil - 3(1) +$$

$$\left\lceil\frac{n+1}{2}\right\rceil - 1 - 3(1) - 1 +$$

$$\left\lceil\frac{n+1}{2}\right\rceil - 3(2) +$$

$$\left\lceil\frac{n+1}{2}\right\rceil - 1 - 3(2) - 1$$

$$+\cdots+\left\lceil\frac{n+1}{2}-\frac{3n_3}{2}\right\rceil$$

Equation 19

Comparing Equation 15 and Equation 19, we realize that a term -1 is added for the odd terms, hence using the sum of integers up to $\left\lceil\frac{q}{2}\right\rceil$, we get for even *n*:

$$P_3(n)=q\left\lceil\frac{n+1}{2}\right\rceil-\frac{3}{2}\left\lfloor\frac{q}{2}\right\rfloor^2-\frac{3}{2}\left\lfloor\frac{q}{2}\right\rfloor-\frac{3}{2}\left\lceil\frac{q}{2}\right\rceil^2-\frac{3}{2}\left\lceil\frac{q}{2}\right\rceil+2\left\lceil\frac{q}{2}\right\rceil-\frac{1}{2}\left[\left(\left\lceil\frac{q}{2}\right\rceil\right)\left(\left\lceil\frac{q}{2}\right\rceil+1\right)\right]$$

Equation 20

Considering Equation 18 and Equation 20, we can use $-(1-|n\ (\mathrm{mod}2)|)$ for the additional term for an even *n*, hence:

$$\begin{aligned}f_3(n)&=P_3(n)+\left\lceil\frac{n+1}{2}\right\rceil\\&=(2a+1)\left\lceil\frac{n+1}{2}\right\rceil-\frac{3}{2}\left(\lfloor a\rfloor^2+\lfloor a\rfloor+\lceil a\rceil^2-\frac{1}{3}\lceil a\rceil\right)\\&\qquad-(1-|n\ (\mathrm{mod}\ 2)|)\frac{(\lceil a\rceil^2+\lceil a\rceil)}{2}\text{, where } a=\frac{\left\lfloor\frac{n}{3}\right\rfloor}{2}\end{aligned}$$

Equation 21

3.3 Examples on proposition 1

(1)

$$P(5) = f_5(5) = P_5(5) + \ f_4(5) = 1 + f_4(5) = 1 + \sum_{i=0}^{\left\lfloor \frac{5}{4} \right\rfloor} f_3(5 - 4i)$$

$$= 1 + f_3(5) + f_3(1) = \ 1 + f_3(5) + 1 = 2 + \sum_{i=0}^{\left\lfloor \frac{5}{3} \right\rfloor} f_2(5 - 3i) = 2 + f_2(5) + f_2(2) =$$

$$= \ 2 + \left\lceil \frac{5+1}{2} \right\rceil + \left\lceil \frac{2+1}{2} \right\rceil = 2 + 3 + 2 = 7$$

Hence the total number of partitions for 5 is 7 (by proposition 1)

(2)

$$P(6) = f_6(6) = P_6(6) + \ f_5(6) = 1 + \sum_{i=0}^{\left\lfloor \frac{6}{5} \right\rfloor} f_4(6 - 5i) = 1 + f_4(6) + f_4(1)$$

$$= 2 + f_4(6) = 2 + \sum_{i=0}^{\left\lfloor \frac{6}{4} \right\rfloor} f_3(6 - 4i) = 2 + f_3(6) + f_3(2) = 2 + 2 + f_3(6)$$

$$= 4 + \sum_{i=0}^{\left\lfloor \frac{6}{3} \right\rfloor} f_2(6 - 3i) = 4 + f_2(6) + f_2(3) + f_2(0) = 5 + \left\lceil \frac{6+1}{2} \right\rceil + \left\lceil \frac{3+1}{2} \right\rceil$$

$$= 4 + 2 + 5 = 11$$

Hence the total number of partitions for 6 is 11 (by proposition 1)

3.4 Examples on proposition 2

(1)

$$f_3(5) = (2a+1)\left\lceil\frac{n+1}{2}\right\rceil - \frac{3}{2}\left(\lfloor a\rfloor^2 + \lfloor a\rfloor + \lceil a\rceil^2 - \frac{1}{3}\lceil a\rceil\right)$$
$$- (1 - |n \ (\mathrm{mod} 2)|)\frac{(\lceil a\rceil^2 + \lceil a\rceil)}{2} \ \text{ where } a = \frac{\left\lfloor\frac{5}{3}\right\rfloor}{2} = 0.5 \text{ and } n = 5$$

$$f_3(5) = 5$$

Check:

5=1+1+1+1+1, 1+1+1+2, 1+2+2, 3+2 and *3+1+1*

5 has 5 partitions comprising 1, 2 and 3 only.

(2)

$$f_3(6) = (2a+1)\left\lceil\frac{n+1}{2}\right\rceil - \frac{3}{2}\left(\lfloor a\rfloor^2 + \lfloor a\rfloor + \lceil a\rceil^2 - \frac{1}{3}\lceil a\rceil\right)$$
$$- (1 - |n \ (\mathrm{mod} 2)|)\frac{(\lceil a\rceil^2 + \lceil a\rceil)}{2} \ \text{ where } a = \frac{\left\lfloor\frac{6}{3}\right\rfloor}{2} = 1 \text{ and } n = 6$$

$$f_3(6) = 7$$

Check:

6=3+3, 3+2+1, 3+1+1+1, 2+2+2, 2+2+1+1, 2+1+1+1+1, 1+1+1+1+1+1

6 has 7 partitions comprising 1, 2 and 3 only.

3.5 Computer Algorithm for proposition 1

We wrote a computer algorithm implementing the formula to calculate the number of partitions using MATLAB:

```
n=input('Please enter the number you wish to partition: ');
a=1;
x=n;
z=n;
b=1;
for count=1:n-2
    a=a+1;
    d=1;
    e=b;
    b=1;
    for count2=1:e % for going to second number in previous row
        for s=0:floor(x(a-1,d)/z) % for using the first number in
previous row
            x(a,b)=x(a-1,d)-(z*s);
            b=b+1;
        end
        d=d+1;
    end
    b=b-1;
    z=z-1;
end
No_of_Partitions=0;
for count3=1:b
    parts=ceil((x(a,count3)+1)/2);
    No_of_Partitions=No_of_Partitions+parts;
end
disp('Number of Partitions is: ')
disp(No_of_Partitions)
```

```
Please enter the number you wish to partition: 1
Number of Partitions is:
     1

Please enter the number you wish to partition: 2
Number of Partitions is:
     2

Please enter the number you wish to partition: 3
Number of Partitions is:
     3

Please enter the number you wish to partition: 4
Number of Partitions is:
     5

Please enter the number you wish to partition: 5
Number of Partitions is:
     7

Please enter the number you wish to partition: 6
Number of Partitions is:
    11

Please enter the number you wish to partition: 7
Number of Partitions is:
    15

Please enter the number you wish to partition: 8
Number of Partitions is:
    22

Please enter the number you wish to partition: 9
Number of Partitions is:
    30

Please enter the number you wish to partition: 10
Number of Partitions is:
    42

Please enter the number you wish to partition: 20
Number of Partitions is:
   627

Please enter the number you wish to partition: 30
Number of Partitions is:
        5604

Please enter the number you wish to partition: 40
Number of Partitions is:
       37338

Please enter the number you wish to partition: 50
Number of Partitions is:
      204226

Please enter the number you wish to partition: 60
Number of Partitions is:
      966467

Please enter the number you wish to partition: 80
```

```
Number of Partitions is:
    15796476
```

n	**Number of Partitions using Proposition 1**
1	1
2	2
3	3
4	5
5	7
6	11
7	15
8	22
9	30
10	42
20	627
30	5604
40	37338
50	204,226
60	966,467
80	15,796,476

CONCLUSION AND FUTURE WORK

In this paper, we presented a brief introduction to the problem of partitioning a positive integer *n*. We presented our method of partitioning a positive integer *n*, and an explicit basic formula for $f_2(n)$ and for $f_3(n)$ are obtained.

During our work on partitioning we used the code in section 3.5 in order to generate a matrix for each positive integer *n*. This matrix is a trace of the algorithm in section 3.5. This algorithm as mentioned before is an implementation of Equation 10, presented under section 3 (Results).

Recalling Equation 10:

$$\boldsymbol{f_m(n) = \sum_{i=1}^{\left\lfloor \frac{n}{m} \right\rfloor} f_{m-1}(n-im) + f_{m-1}(n) = \sum_{i=0}^{\left\lfloor \frac{n}{m} \right\rfloor} f_{m-1}(n-im)}$$

For example, to find the number of partitions of 6 using the above equation:

$$f_6(6) = \sum_{i=0}^{\left\lfloor \frac{6}{6} \right\rfloor} f_5(6-6i) = f_5(6) + f_5(0)$$

$$= \sum_{i=0}^{\left\lfloor \frac{6}{5} \right\rfloor} f_4(6-5i) + \sum_{i=0}^{\left\lfloor \frac{0}{5} \right\rfloor} f_4(0-5i) = f_4(6) + f_4(1) + f_4(0)$$

$$= \sum_{i=0}^{\left\lfloor \frac{6}{4} \right\rfloor} f_3(6-4i) + \sum_{i=0}^{\left\lfloor \frac{1}{4} \right\rfloor} f_3(1-4i) = f_3(6) + f_3(2) + f_3(1) + f_3(0)$$

$$= \sum_{i=0}^{\left\lfloor \frac{6}{3} \right\rfloor} f_2(6-3i) + \sum_{i=0}^{\left\lfloor \frac{2}{3} \right\rfloor} f_2(2-3i) + \sum_{i=0}^{\left\lfloor \frac{1}{3} \right\rfloor} f_{z-1}(1-3i)$$

$$= f_2(6) + f_2(3) + f_2(0) + f_2(2) + f_2(1) + f_2(0)$$

$$= \left\lceil \frac{6+1}{2} \right\rceil + \left\lceil \frac{3+1}{2} \right\rceil + \left\lceil \frac{0+1}{2} \right\rceil + \left\lceil \frac{2+1}{2} \right\rceil + \left\lceil \frac{1+1}{2} \right\rceil + \left\lceil \frac{0+1}{2} \right\rceil = 4 + 2 + 1 + 2 + 1 + 1 = 11$$

The first row in the matrix generated by setting *n =6* will contain the number itself. The second row will contain the entries in $f_5(n)$. So as seen above, for number 6 the entries are 6 and 0.

The second row will contain the entries in $f_4(n)$. So as seen above, for number 6 the entries are 6, 1 and 0.

Until reaching the last row of the matrix that represents the numbers that shall be placed in $f_2(n)$ in order to calculate the total number of partitions of n.

For example, take the number 6. As it is shown below, the last row in the generated matrix associated to the number 6 is: 6 3 2 1 0 0. Hence

$$P(6) = f_6(6) = f_2(6) + f_2(3) + f_2(2) + f_2(1) + f_2(0) + f_2(0) = 11$$

However, a pattern can be noticed after writing down several numbers.

For 1:

1

For 2:

2

For 3:

3	0
3	**0**

For4:

4	0	0
4	0	0
4	**1**	**0**

For5:

5	0	0	0
5	0	0	0
5	1	0	0
5	**2**	**1**	**0**

For 6:

6	0	0	0	0	0
6	0	0	0	0	0
6	1	0	0	0	0
6	2	1	0	0	0
6	3	0	2	1	0

6 3 2 1 0 0

For 7:

7	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0
7	1	0	0	0	0	0	0

7	2	1	0	0	0	0	0
7	3	2	1	0	0	0	0
7	4	1	3	0	2	1	0

7 4 3 2 1 1 0 0

For 8:

8	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
8	1	0	0	0	0	0	0	0	0	0
8	2	1	0	0	0	0	0	0	0	0
8	3	2	1	0	0	0	0	0	0	0
8	4	0	3	2	1	0	0	0	0	0
8	5	2	4	1	0	3	0	2	1	0

8 5 4 3 2 2 1 1 0 0 0

For 9:

9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	1	0	0	0	0	0	0	0	0	0	0	0	0	0
9	2	1	0	0	0	0	0	0	0	0	0	0	0	0
9	3	2	1	0	0	0	0	0	0	0	0	0	0	0
9	4	3	2	1	0	0	0	0	0	0	0	0	0	0
9	5	1	4	0	3	2	1	0	0	0	0	0	0	0
9	6	3	0	5	2	1	4	1	0	3	0	2	1	0

9 6 5 4 3 3 2 2 1 1 1 0 0 0 0

For 10:

10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	3	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	4	3	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	5	0	4	3	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0
10	6	2	5	1	0	4	0	3	2	1	0	0	0	0	0	0	0	0	0
10	7	4	1	6	3	0	2	5	2	1	0	4	1	0	3	0	2	1	0

10 7 6 5 4 4 3 3 2 2 2 1 1 1 1 0 0 0 0 0

Table 2

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
1		1																			
2			1																		
3	1			1																	
4	1	1			1																
5	1	1	1			1															
6	2	1	1	1			1														
7	2	2	1	1	1			1													
8	3	2	2	1	1	1			1												
9	4	3	2	2	1	1	1			1											
10	5	4	3	2	2	1	1	1			1										
11	6	5	4	3	2	2	1	1	1			1									
12	9	6	5	4	3	2	2	1	1	1			1								
13	10	9	6	5	4	3	2	2	1	1	1			1							
14	13	10	9	6	5	4	3	2	2	1	1	1			1						
15	17	13	10	9	6	5	4	3	2	2	1	1	1			1					
16	21	17	13	10	9	6	5	4	3	2	2	1	1	1			1				
17	25	21	17	13	10	9	6	5	4	3	2	2	1	1	1			1			
18	23	25	21	17	13	10	9	6	5	4	3	2	2	1	1	1			1		
19	39	23	25	21	17	13	10	9	6	5	4	3	2	2	1	1	1			1	
20	49	39	23	25	21	17	13	10	9	6	5	4	3	2	2	1	1	1			1

In order to calculate the number of occurrence of each number, we wrote the algorithm in Appendix 1.

As shown in Table 2, there is a pattern in the numbers of occurrence of each number that when placed in $f_2(n)$ gives the total number of partitions of that number.

Figure 1 below shows the zeros occurrences using the code attached in Appendix IV.

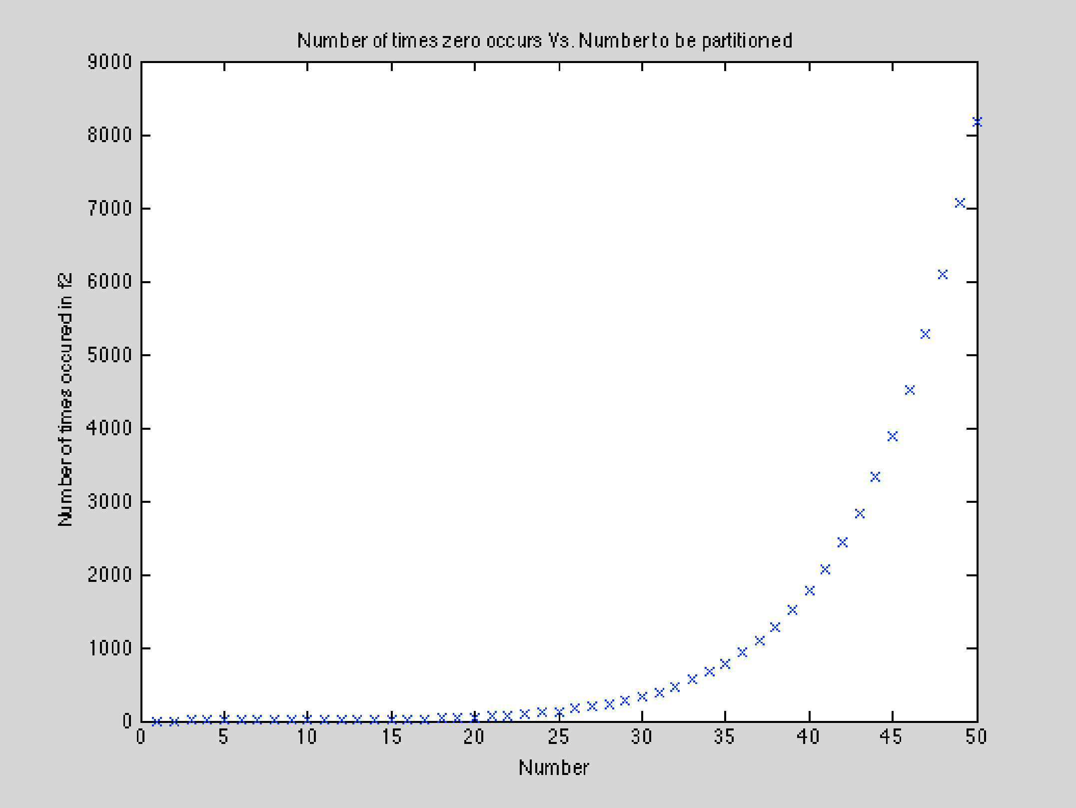


Figure 1: Number of zeros occurrence Vs. number to be partitioned

Considering the case where $8 \leq n \leq 11$:

$$p(8) = p(7) - f_2(7) + f_2(8) + f_2(5) + f_2(2) + f_2(0)$$

For n=9:

$$p(9) = p(8) - f_2(8) + f_2(9) + f_2(6) + f_2(3) + f_2(0) + f_2(0)$$

And the same pattern would continue as we move on until we reach 11, because always the term having the zero would increase by one, and the function can be extracted from the previous expressions that would yield:

$$p(n) = p(n-1) - f_2(n-1) + f_2(n) + f_2(n-3) + f_2(n-6) + \sum_{k=0}^{n-8} f_2(k)$$

$$\text{for } 8 \leq n \leq 11$$

Simplifying the previous expression yields:

$$p(n) = p(n-1) - f_2(n-1) + f_2(n) + f_2(n-3) + f_2(n-6) + \sum_{k=0}^{n-8} f_2(k)$$

$$p(n) = p(n-1) - \left\lceil \frac{n-1+1}{2} \right\rceil + f_2(n) + \left\lceil \frac{n-3+1}{2} \right\rceil + \left\lceil \frac{n-6+1}{2} \right\rceil + \sum_{k=0}^{n-8} f_2(k)$$

$$p(n) = p(n-1) - \left\lceil \frac{n}{2} \right\rceil + f_2(n) + \left\lceil \frac{n}{2} - 1 \right\rceil + \left\lceil \frac{n+1}{2} - 3 \right\rceil + \sum_{k=0}^{n-8} f_2(k)$$

$$p(n) = p(n-1) - \left\lceil \frac{n}{2} \right\rceil + f_2(n) + \left\lceil \frac{n}{2} \right\rceil - 1 + \left\lceil \frac{n+1}{2} \right\rceil - 3 + \sum_{k=0}^{n-8} f_2(k)$$

$$p(n) = p(n-1) - \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil - 1 - 3 + f_2(n) + \left\lceil \frac{n+1}{2} \right\rceil + \sum_{k=0}^{n-8} f_2(k)$$

$$p(n) = p(n-1) - 4 + 2f_2(n) + \sum_{k=0}^{n-8} f_2(k) \quad \text{for} \quad 8 \le n \le 11$$

Equation 22

Plotting the Number of partitions Versus the number to be partitioned we obtain the following graph:

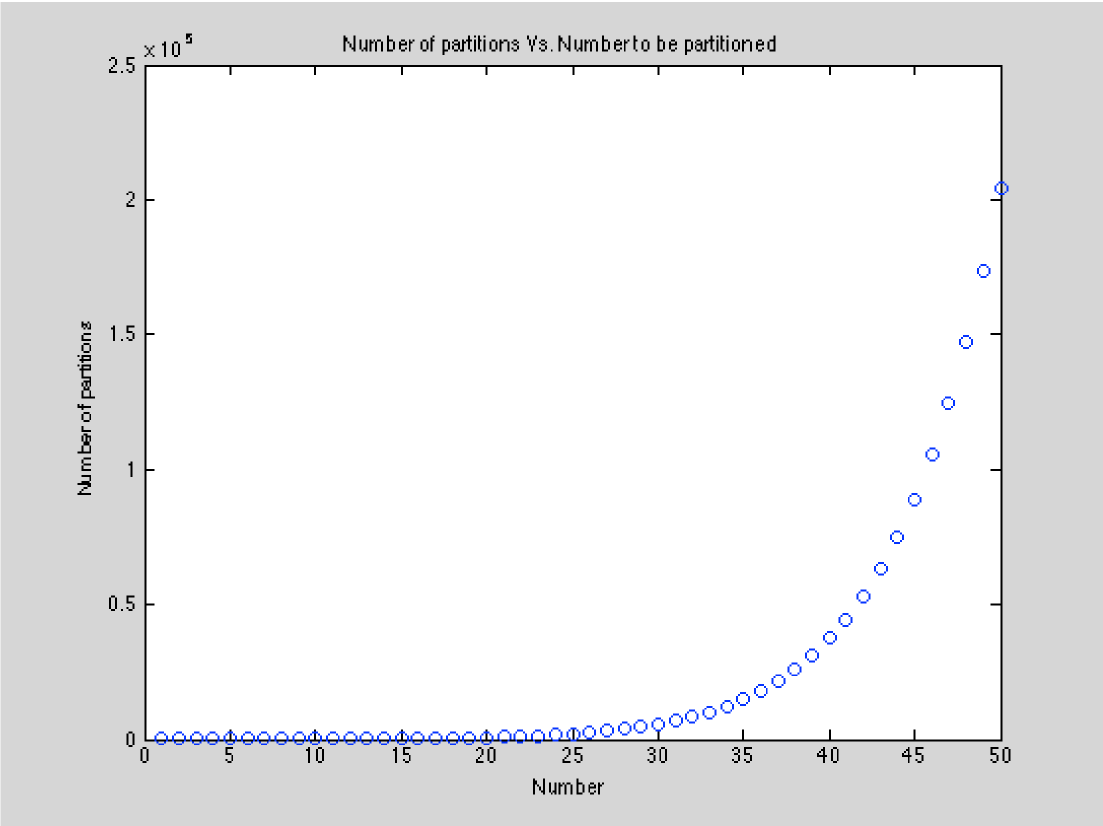


Figure 2: Number of Partitions Vs. number to be partitioned

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APPENDIX

I

```
%% to count the number of occurrence of each number in the final f2
count=0;
[a,b]=size(x);
for i=1:b
    if x(a,i)==0
        count=count+1;
    end
end
n
count
```

II

```
%% to get the sequence where we get 0 in the f2
[a,b]=size(x);
c=x(a);
d=1;
for i=1:b
    if x(a,i)==0
        c(d,1)=i;
        d=d+1;
    end
end
c
```

III

```
%% to get the difference between each row and the other

for i=1:size(c)-1
    e(i)=c(i+1)-c(i);
end
e'
```

IV

```
%% To get the graph of the number of zeros

no_of_zeros=[0 0 1 1 1 2 2 3 4 5 6 9 10 13 17 21 25 33 39 49 60 73 88 110
130 158 191 230 273 331 391 468 556 660 779 927 1087 1284 1510 1775 2075
2438 2842 3323 3872 4510 5273 6095 7056 8182];
no_of_parts=[
           1
           2
           3
           5
           7
          11
          15
          22
          30
          42
          56
          77
         101
         135
         176
         231
         297
         385
         490
         627
```

```
            792
           1002
           1255
           1575
           1958
           2436
           3010
           3718
           4565
           5604
           6842
           8349
          10143
          12310
          14883
          17977
          21637
          26015
          31185
          37338
          44583
          53174
          63261
          75175
          89134
         105558
         124754
         147273
         173525
         204226];
no_of_ones=[1
              0
              0
              1
              1
              1
              2
              2
              3
              4
              5
              6
              9
             10
             13
             17
             21
             25
             33
             39
             49
             60
             73
             88
            110
            130
            158
            191
            230
            273
            331
            391
            468
            556
            660
            779
            927
           1087
```

```
        1284
        1510
        1775
        2075
        2438
        2842
        3323
        3872
        4510
        5237
        6095
        7056]';
plot(no_of_zeros,'x');

hold off
```